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On Kergin Interpolation in the Disk

LEN BOS

Department of Mathematics,
University of Toronto, Toronto, Canada

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1. INTRODUCTION

Kergin [3] introduced a natural extension of the Newton form of single variable interpolation to the case of several variables. An explicit formula is given by Micchelli and Milman [5]. The main result of this paper is that if $K_{n,m}(x,y)$ is the Kergin interpolant of degree $n+m-1$ to the function $x^n y^m$ at the $n+m$ points $(\cos 2k\pi/(n+m), \sin 2k\pi/(n+m))$, $1 \leqslant k \leqslant n+m$, and we set

$$P_{n,m} = 2^{n+m-1} \binom{n+m}{m}(x^n y^m - K_{n,m}) + \binom{n+m}{m} \begin{cases} 1, & m \equiv 0\ (4), \\ 0, & m \equiv 1\ (4), \\ -1, & m \equiv 2\ (4), \\ 0, & m \equiv 3\ (4), \end{cases}$$

then the $P_{n,m}$ satisfy the recurrence relation

$$P_{n,m} = 2xP_{n-1,m} + 2yP_{n,m-1} - P_{n-2,m} - P_{n,m-2}$$

with $P_{0,0} = 1$, $P_{1,0} = x$, $P_{0,1} = y$, and hence are the "Chebyshev polynomials" for the disk studied by Reimer [6].

Reimer has shown that, in fact, $|P_{n,m}| \leqslant \binom{n+m}{m}$ on $x^2 + y^2 \leqslant 1$ and we thus obtain the immediate corollary that

$$\begin{aligned} |x^n y^m - K_{n,m}(x,y)| &\leqslant 2^{-(n+m-1)}, \qquad m \text{ odd}, \\ &\leqslant 2^{-(n+m-2)}, \qquad m \text{ even}. \end{aligned}$$

Further, we use the properties of Kergin interpolation to derive the property

$$\frac{\partial^k P_{n,m+k}}{\partial y^k} = \frac{\partial^k P_{n+k,m}}{\partial x^k}.$$

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Finally, an explicit formula

$$P_{n,m}(x,y) = \sum_{k=0}^{[m/2]} (-1)^k \frac{n+m}{(n+m-k)\, k!\, 2^k (m-2k)!} \times y^{m-2k} T_{n+m-k}^{(m-k)}(x),$$

where $T_j(x) = \cos(j \cos^{-1} x)$ is the jth Chebyshev polynomial, is given.

2. Kergin Interpolation

We give the definition and basic properties of Kergin interpolation using the approach of Micchelli and Milman [5].

Let S_N denote the N-simplex, $S_N = \{(\varepsilon_0, \varepsilon_1, ..., \varepsilon_N) : \varepsilon_i \geqslant 0, \sum \varepsilon_i = 1\}$ and, for any sequence of $N+1$ points, $x_0, x_1, ..., x_N \in \mathbb{R}^n$, and continuous function $g : \mathbb{R}^n \to \mathbb{R}$, set

$$\int_{[x_0, \ldots, x_N]} g = \int_{S_N} g \left(\sum_{k=0}^{N} \varepsilon_k x_k \right) d\varepsilon_1 d\varepsilon_2 \cdots d\varepsilon_N.$$

Define maps $\pi_m : C^m(\mathbb{R}^n) \to \mathbb{P}_m(\mathbb{R}^n)$, the polynomials of degree at most m, by

$$(\pi_m f)(x) = \left(\int_{[x_0, \ldots, x_m]} d^m f \right)(x - x_0, x - x_1, ..., x - x_{m-1}),$$

where $d^m f$ is the mth total derivative of f.

We note that in one variable, by the Hermite–Genocchi formula,

$$(\pi_m f)(x) = f[x_0, x_1, ..., x_m](x - x_0, x - x_1, ..., x - x_{m-1}),$$

where $f[x_0, x_1, ..., x_m]$ is the mth divided difference of f at the given points.

Finally, define the map $K : C^N(\mathbb{R}^n) \to \mathbb{P}_N(\mathbb{R}^n)$ by

$$K = \sum_{m=0}^{N} \pi_m.$$

Then Kf is the Kergin interpolant to f at $(x_i)_0^N$. We note that, as an operator, K is linear and continuous. In one variable Kf provides the Newton form of the interpolating polynomial.

Remark 2.1. If $f = g \circ \lambda$ for some g in $C^N(\mathbb{R})$ and some linear map $\lambda : \mathbb{R}^n \to \mathbb{R}$, then Kf is the one-variable polynomial which interpolates g at the points $(\lambda(x_i))_0^N$, composed with λ.

THEOREM 2.2 ([5]). *If q is a homogeneous polynomial of degree k, $0 \leqslant k \leqslant N$, then*

$$\int_{[x_0, \ldots, x_k]} q(\partial/\partial x)(Kf - f) = 0.$$

THEOREM 2.3 ([3]). *If P is a polynomial of degree N that has the property of Kf in Theorem 2.2 for any ordering of the points $(x_i)_0^N$, then $P = Kf$.*

THEOREM 2.4 (*Milman and Micchelli* [4]). *If f is in $C^{N+1}(\mathbb{R}^n)$, then*

$$(f - Kf)(x) = \left(\int_{[x_0, \ldots, x_N, x]} d^{N+1} f\right)(x - x_0, \ldots, x - x_N).$$

COROLLARY. *The map K is a projector.*

THEOREM 2.5. *If p and q are homogeneous polynomials of degree k, $0 \leqslant k$, and $p(\partial/\partial x) f = q(\partial/\partial x) g$ for some functions f, g in $C^N(\mathbb{R}^n)$, then $p(\partial/\partial x) Kf = q(\partial/\partial x) Kg$.*

Proof. If $k > N$, both sides are zero, so we assume that $k \leqslant N$. Suppose that $Kf = P_0 + P_1 + \cdots + P_N$ and $Kg = Q_0 + Q_1 + \cdots + Q_N$ are the homogeneous decompositions of Kf and Kg, respectively. We shall show that $p(\partial/\partial x) P_j = q(\partial/\partial x) Q_j$, $0 \leqslant j \leqslant N$.

Using usual multi-index notation, note that, by Theorem 2.2,

$$\int_{[x_0, x_1, \ldots, x_{k+|i|}]} \frac{\partial^{|i|}}{\partial x^i} p(\partial/\partial x)(f - Kf)$$
$$= \int_{[x_0, x_1, \ldots, x_{k+|i|}]} \frac{\partial^{|i|}}{\partial x^i} q(\partial/\partial x)(g - Kg) = 0$$

for $|i| \leqslant N - k$. Subtracting, we see that

$$\int_{[x_0, x_1, \ldots, x_{k+|i|}]} \frac{\partial^{|i|}}{\partial x^i} (p(\partial/\partial x) Kf - q(\partial/\partial x) Kg) = 0.$$

Hence there is some point x such that

$$\frac{\partial^{|i|}}{\partial x^i} (p(\partial/\partial x) Kf - q(\partial/\partial x) Kg) = 0.$$

Now consider $j=N$. For $|i|=N-k$,

$$\frac{\partial^{|i|}}{\partial x^i}(p(\partial/\partial x)P_N - q(\partial/\partial x)Q_N)$$

$$= \frac{\partial^{|i|}}{\partial x^i}(p(\partial/\partial x)Kf - q(\partial/\partial x)Kg) = 0$$

at some point x. The first equality follows from the fact that lower degree terms are differentiated away and the second by the above remark. Hence $p(\partial/\partial x)P_N - q(\partial/\partial x)Q_N$ is a homogeneous polynomial of degree $N-k$, all of whose $(N-k)$th order partials vanish at some point. It is, therefore, identically zero.

Now consider $k \leqslant j < N$ and assume that for $t > j$, $p(\partial/\partial x)P_t - q(\partial/\partial x)Q_t \equiv 0$. Then for $|i| = j-k$, by this hypothesis,

$$\frac{\partial^{|i|}}{\partial x^i}(p(\partial/\partial x)P_j - q(\partial/\partial x)Q_j)$$

$$= \frac{\partial^{|i|}}{\partial x^i}(p(\partial/\partial x)Kf - q(\partial/\partial x)Kg).$$

Again, this last expression is zero at some point and, as before, $p(\partial/\partial x)P_j - q(\partial/\partial x)Q_j \equiv 0$. The result follows by reverse induction. ∎

3. Kergin Interpolation at Equally Spaced Points on the Unit Circle

As before, let $K_{n,m}$ be the Kergin polynomial interpolating $x^n y^m$ at the $n+m$ points $(\cos 2k\pi/(n+m),\ \sin 2k\pi/(n+m))$, $1 \leqslant k \leqslant n+m$. An examination of the formula of Theorem 2.4 reveals that

$$\binom{n+m}{m}(x^n y^m - K_{n,m}(x,y)) = \sum_{\substack{S \subset \{1,2,\ldots,n+m\} \\ |S|=n}} \prod_{k\in S}(x-\cos\theta_k) \prod_{k\notin S}(y-\sin\theta_k), \qquad (3.1)$$

where we have set $\theta_k = 2k\pi/(n+m)$. It is surprising that such a formidable expression has pleasant properties. We set

$$P_{n,m}(x,y) = 2^{(n+m-1)}\binom{n+m}{m}(x^n y^m - K_{n,m}(x,y)) + \binom{n+m}{m} \times \begin{cases} 1, m \equiv 0\ (4), \\ 0, m \equiv 1\ (4), \\ -1, m \equiv 2\ (4), \\ 0, m \equiv 3\ (4), \end{cases} \tag{3.2}$$

and calculate the generating function of these polynomials.

LEMMA 3.3. *If* $\mathbf{t} = (t_1, t_2)$, *then*

$$\sum_{n+m=d} P_{n,m}(x,y)\, t_1^n t_2^m = |\mathbf{t}|^d\, T_d((xt_1 + yt_2)/|\mathbf{t}|).$$

Proof. We first compute $\sum_{n+m=d} 2^{(n+m-1)}\binom{n+m}{m}(x^n y^m - K_{n,m}(x,y))\, t_1^n t_2^m$. By linearity, this expression equals

$$2^{d-1} \sum_{n+m=d} \binom{d}{n}(xt_1)^n (yt_2)^m - K\left(\binom{d}{n}(xt_1)^n (yt_2)^m\right)(x,y)$$
$$= 2^{d-1}((xt_1 + yt_2)^d - K((xt_1 + yt_2)^d)(x,y)).$$

We now apply Remark 2.1 to obtain

$$2^{d-1} \prod_{k=1}^{d} ((xt_1 + yt_2) - (t_1 \cos\theta_k + t_2 \sin\theta_k))$$
$$= 2^{d-1} |\mathbf{t}|^d \prod_{k=1}^{d} ((xt_1 + yt_2)/|\mathbf{t}| - (t_1 \cos\theta_k + t_2 \sin\theta_k)/|\mathbf{t}|))$$
$$= 2^{d-1} |\mathbf{t}|^d \prod_{k=1}^{d} (\cos\phi - (\cos\theta \cos\theta_k + \sin\theta \sin\theta_k)).$$

Here, we have set $\cos\phi = (xt_1 + yt_2)/|\mathbf{t}|$, $\cos\theta = t_1/|\mathbf{t}|$, and $\sin\theta = t_2/|\mathbf{t}|$.

Clearly, this last expression is equal to

$$2^{d-1} |\mathbf{t}|^d \prod_{k=1}^{d} (\cos\phi - \cos(\theta - \theta_k)),$$

which, for brevity, we refer to as $Q(\phi)$. Then

$$Q(\phi - \theta) = 2^{d-1} |\mathbf{t}|^d \prod_{k=1}^{d} (\cos(\theta - \phi) - \cos(\theta - \theta_k))$$
$$= |\mathbf{t}|^d (\cos d(\theta - \phi) - \cos d\theta),$$

both sides being polynomials in $\cos(\theta-\phi)$ with same degrees, zeros, and leading coefficients. Hence

$$Q(\phi)=|\mathbf{t}|^d\,(\cos d\phi-\cos d\theta),$$

and our original sum is

$$|\mathbf{t}|^d\,(T_d((xt_1+yt_2)/|\mathbf{t}|)-T_d(t_1/|\mathbf{t}|)).$$

Further,

$$\sum_{n+m=d} t_1^n t_2^m \binom{d}{n} \begin{cases} 1, m\equiv 0\ (4), \\ 0, m\equiv 1\ (4), \\ -1, m\equiv 2\ (4), \\ 0, m\equiv 3\ (4), \end{cases}$$

$$\begin{aligned} &=\operatorname{Re}(t_1+it_2)^d \\ &=|\mathbf{t}|^d\operatorname{Re}(t_1/|\mathbf{t}|+it_2/|\mathbf{t}|)^d \\ &=|\mathbf{t}|^d\operatorname{Re}(\cos\theta+i\sin\theta)^d, \end{aligned}$$

where we have set $\cos\theta=t_1/|\mathbf{t}|$ and $\sin\theta=t_2/|\mathbf{t}|$.

By de Moivre's theorem, this simplifies to

$$|\mathbf{t}|^d\cos d\theta=|\mathbf{t}|^d\,T_d(t_1/|\mathbf{t}|).$$

The result follows from the addition of the two sums. ∎

An immediate consequence of this calculation is that the generating function satisfies

$$\begin{aligned} \sum_{n,m=0}^{\infty} P_{n,m}t_1^n t_2^m &= \sum_{d=0}^{\infty} |\mathbf{t}|^d\,T_d((xt_1+yt_2)/|\mathbf{t}|) \\ &=(1-(xt_1+yt_2))/(1-2(xt_1+yt_2)+|\mathbf{t}|^2). \end{aligned}$$

We have made use of the fact that the generating function of the Chebyshev polynomials is known to be

$$\sum_{k=0}^{\infty} T_k(x)\,t^k=(1-xt)/(1-2xt+t^2).$$

It follows from the generating function that the $P_{n,m}$ satisfy the recurrence relation

$$P_{n,m} = 2xP_{n-1,m} + 2yP_{n,m-1} - P_{n-2,m} - P_{n,m-2},$$
$$P_{0,0} = 1, \qquad P_{1,0} = x, \qquad \text{and} \qquad P_{0,1} = y.$$

The polynomials determined by this relation were studied by Reimer [6]. He proves the following two theorems:

THEOREM 3.4 ([6]). *For* $x^2 + y^2 \leqslant 1$, $|P_{n,m}(x, y)| \leqslant \binom{n+m}{m}$.

THEOREM 3.5 ([6]). *Among all polynomials of the form* $Q_{n,m}(x, y) = 2^{(n+m-1)}\binom{n+m}{m} x^n y^m +$ *(lower degree terms),* $\max_{x^2+y^2 \leqslant 1} |Q_{n,m}(x, y)|$ *is least for* $Q_{n,m} = P_{n,m}$.

The following interesting identity is made use of in the proof of Theorem 3.5:

LEMMA 3.6 ([6]). *We have*

$$P_{n,m}(\cos\theta, \sin\theta) = \binom{n+m}{m} \begin{cases} \cos(m+n)\,\theta, & m \equiv 0\ (4), \\ \sin(m+n)\,\theta, & m \equiv 1\ (4), \\ -\cos(m+n)\,\theta, & m \equiv 2\ (4), \\ -\sin(m+n)\,\theta, & m \equiv 3\ (4). \end{cases}$$

Now by formula (3.2) for $P_{n,m}$, the following is immediate.

THEOREM 3.7. *For* $x^2 + y^2 \leqslant 1$,

$$\begin{aligned} |x^n y^m - K_{n,m}(x, y)| &\leqslant 2^{-(n+m-1)}, \qquad m \text{ odd}, \\ &\leqslant 2^{-(n+m-2)}, \qquad m \text{ even}. \end{aligned}$$

Also, for more general f, we substitute (3.1) into the error formula of Theorem 2.4 to obtain

THEOREM 3.8. *If f is in* $C^{N+1}(\mathbb{R}^n)$ *and Kf is the Kergin interpolatnt to f at the $N+1$ points* $\mathbf{x}_k = (\cos 2k\pi/(N+1), \sin 2k\pi/(N+1))$, $0 \leqslant k \leqslant N$, *then, on the unit disk,*

$$|(f - Kf)(\mathbf{x})| \leqslant \frac{1}{(N+1)!\, 2^{N-1}} \sum_{\alpha+\beta=N+1} \binom{N+1}{\alpha} \left\| \frac{\partial^{N+1} f}{\partial x^\alpha\, \partial y^\beta} \right\|_\infty.$$

Proof. By Theorem 2.4,

$$
\begin{aligned}
&|(f-Kf)(\mathbf{x})| \\
&= \left| \left(\int_{[\mathbf{x}_0,\ldots,\mathbf{x}_N,\mathbf{x}]} d^{N+1} f \right)(\mathbf{x}-\mathbf{x}_0, \mathbf{x}-\mathbf{x}_1, \ldots, \mathbf{x}-\mathbf{x}_N) \right| \\
&= \left| \sum_{\alpha+\beta=N+1} \left(\int_{[\mathbf{x}_0,\ldots,\mathbf{x}_N,\mathbf{x}]} \frac{\partial^{N+1} f}{\partial x^\alpha \partial y^\beta} \right) \right. \\
&\qquad \left. \times \sum_{S \subset \{0,1,\ldots,N\}} \prod_{k \in S} (x - \cos\theta_k) \prod_{k \notin S} (y - \sin\theta_k) \right| \\
&\leqslant \left| \sum_{\alpha+\beta=N+1} \binom{N+1}{\alpha} 2^{-(N-1)} \left\| \frac{\partial^{N+1} f}{\partial x^\alpha \partial y^\beta} \right\|_\infty \int_{[\mathbf{x}_0,\ldots,\mathbf{x}_N,\mathbf{x}]} 1 \right| .
\end{aligned}
$$

The result follows from the computation

$$\int_{[\mathbf{x}_0,\ldots,\mathbf{x}_N,\mathbf{x}]} 1 = 1/(N+1)!. \quad \blacksquare$$

Now, by noticing that $(\partial/\partial y^k)\binom{n+m+k}{n} x^n y^{m+k} = (\partial/\partial x^k)\binom{n+m+k}{m} x^{n+k} y^m$ and applying Theorem 2.5, it follows immediately that

PROPOSITION 3.9. *We have*

$$\frac{\partial}{\partial y^k} P_{n,m+k} = \frac{\partial}{\partial x^k} P_{n+k,m}.$$

Our last result is an explicit formula for $P_{n,m}(x, y)$.

THEOREM 3.10. *We have*

$$
\begin{aligned}
&P_{n,m}(x,y) \\
&= \sum_{k=0}^{[m/2]} (-1)^k \frac{m+n}{(n+m-k)\, 2^k k!\, (m-2k)!} y^{m-2k} T_{n+m-k}^{(m-k)}(x) \\
&= \sum_{k=0}^{[n/2]} (-1)^k \frac{n+m}{(m+n-k)\, 2^k k!\, (n-2k)!} x^{n-2k} T_{m+n-k}^{(n-k)}(y).
\end{aligned}
$$

Proof. We prove the first formula; the second follows from the fact that $P_{n,m}(x, y) = P_{m,n}(y, x)$. We proceed by induction on m.

For $m = 0$, the right side reduces to $T_n(x)$. But $P_{n,0}(x, y)$ satisfies

$$P_{n,0}(x,y) = 2xP_{n-1,0}(x,y) - P_{n-2,0}(x,y), \qquad P_{0,0} = 1, \quad P_{1,0} = x$$

and hence also equals $T_n(x)$.

Now assume that the equation holds for fixed m and all n. Then

$$\begin{aligned}\frac{\partial P_{n,m+1}}{\partial y} &= \frac{\partial P_{n+1,m}}{\partial x}\\ &= \frac{\partial}{\partial x}\sum_{k=0}^{[m/2]}(-1)^k\frac{m+n+1}{(n+1+m-k)\,2^k k!\,(m-2k)!}\\ &\quad\times y^{m-2k}T^{(m-k)}_{m+n+1-k}(x)\\ &= \sum_{k=0}^{[m/2]}(-1)^k\frac{m+n+1}{(n+1+m-k)\,2^k k!(m-2k)!}\\ &\quad\times y^{m-2k}T^{(m+1-k)}_{n+m+1-k}(x)\\ &= \frac{\partial}{\partial y}\sum_{k=0}^{[(m+1)/2]}(-1)^k\frac{m+n+1}{(n+m+1-k)\,2^k k!\,(m+1-2k)!}\\ &\quad\times y^{m+1-2k}T^{(m+1-k)}_{n+m+1-k}(x).\end{aligned}$$

Hence $P_{n,m+1}$ and the corresponding expressions differ by a polynomial in x alone. That they are identically equal follows from the computational lemma below. ∎

LEMMA 3.11. *We have*

$$\begin{aligned}P_{n,m}(x,0) &= 0, && \text{if } m \text{ odd},\\ &= [(-1)^k\,(n+2k)/2^k k!\,(n+k)]\,T^{(k)}_{n+k}(x), && \text{if } m=2k.\end{aligned}$$

Proof. The case of m odd follows immediately from the recurrence relation. For even m, we use induction on $n+m$. The cases $n+m=0, 1$ are immediate. Now it is known that (see, e.g., Rivlin [7, p. 32]).

$$T_r(x)=\sum_{j=0}^{[r/2]}(-1)^j\,r/(r-j)\binom{r-j}{j}\,2^{r-2j-1}x^{r-2j}.$$

Thus,

$$\begin{aligned}&\frac{(-1)^k\,(n+2k)}{(n+k)\,2^k k!}T^{(k)}_{n+k}(x)\\ &\quad=(-1)^k\sum_{j=0}^{[n/2]}(-1)^j\frac{n+2k}{n+k-j}\\ &\qquad\times\binom{n+k-j}{j}\binom{n+k-2j}{k}\,2^{n-2j-1}x^{n-2j},\end{aligned}\tag{3.3}$$

$$\frac{(2x)(-1)^k (n-1+2k)}{(n-1+k)\, 2^k k!} T^{(k)}_{n-1+k}(x)$$
$$= (-1)^k \sum_{j=0}^{[(n-1)/2]} (-1)^j \frac{n-1+2k}{n-1+k-j}$$
$$\times \binom{n-1+k-j}{j} \binom{n-1+k-2j}{k} 2^{n-2j-1} x^{n-2j}, \qquad (3.4)$$

$$\frac{(-1)^{k+1} (n-2+2k)}{(n-2+k)\, 2^k k!} T^{(k)}_{n-2+k}(x)$$
$$= (-1)^{k+1} \sum_{j=0}^{[(n-2)/2]} (-1)^j \frac{n-2+2k}{n-2+k-j}$$
$$\times \binom{n-2+k-j}{j} \binom{n-2+k-2j}{k} 2^{n-2-2j-1} x^{n-2-2j}, \qquad (3.5)$$

and

$$\frac{(-1)^k (n+2(k-1))}{(n+k-1)\, 2^{k-1}(k-1)!} T^{(k-1)}_{n+k-1}(x)$$
$$= (-1)^k \sum_{j=0}^{[n/2]} (-1)^j \frac{n-2+2k}{n+k-1-j}$$
$$\times \binom{n+k-1-j}{j} \binom{n+k-1-2j}{k-1} 2^{n-2j-1} x^{n-2j}. \qquad (3.6)$$

By comparison of coefficients we see that (3.3) = (3.4) + (3.5) + (3.6) and hence the given formula satisfies the recurrence relation. The result follows. ∎

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